were used for calculating K. With this value of K, the values of

$$\int_{x_1}^{x_n} \frac{St}{K}\, dx$$

are somewhat higher than the values of

$$\int_{x_1}^{x_n} \frac{C_f}{2}\, dx$$

as shown in Fig. 2. It is of interest to note that, when $K = 1$, then the values of

$$\int_{x_1}^{x_n} \frac{St}{K}\, dx$$

are nearly the same as the values of Ref. 2. An empirical value for K for the zero-pressure gradient case has been found by Colburn,[3] which is simply $K = Pr^{-2/3}$. Using Colburn's K, the values of

$$\int_{x_1}^{x_n} \frac{St}{K}\, dx$$

obtained are in good agreement with the integrated values of the skin-friction coefficient.

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Radiation Slip

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It long has been recognized (see, e.g., Goulard[1]) that a close analogy exists between radiative transfer processes, as ascribed to the motion of a photon gas, and molecular transport phenomena. In analyzing radiative heat transfer between a gas and a solid surface, it soon is apparent that the analogy to low-density conductive heat transfer is sufficiently close that many of the concepts should be interchangeable. On this basis, we propose that the difficult problem of calculating radiative heat transfer between a gas and a surface in the "transition" regime, between an optically thick (opaque) medium and an optically thin (transparent) one, may be circumvented by the use of a simple rarefied flow model. The model uses the radiation conduction equation, valid for an optically dense medium (Rosseland or diffusion approximation[2]), in conjunction with what we shall term a "radiation slip" boundary condition.

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The radiation slip condition is analogous to the Maxwell velocity slip in a low-density molecular gas, where there is a discontinuity at the surface between the gas and wall velocity. It is, however, more specifically analogous to the Smoluchowski temperature jump, in which a discontinuity between the gas and wall temperature exists in a rarefied molecular medium. The proposed boundary condition may be stated: in an absorbing and emitting radiating medium there is, adjacent to a solid surface, a temperature discontinuity that is proportional to the local photon mean free path multiplied by the radiation temperature gradient.

To illustrate the present idea, the calculation of the radiative heat transfer for an absorbing-emitting gas contained between infinite parallel walls separated by a distance Δ and maintained at different uniform temperatures is considered (see Fig. 1). In order to bring out the main features of the present approach, at first the gas will be treated as nonconducting, in the sense that it does not transfer heat by molecular conduction. Furthermore, both walls are taken to be blackbodies so that they are perfect absorbers of any incident radiation. The problem is to determine, say, for a fixed spacing between the plates, how the net radiative heat transfer rate to the cooler wall changes as the photon mean free path in the gas is varied. The photon mean free path l_ν is defined here as the inverse of the volumetric absorption coefficient frequently used in radiative engineering calculations. The nature of the radiative transfer then will be characterized by the appropriate dimensionless optical length τ, which, for the present problem, is

$$\tau = \Delta / l_\nu \tag{1}$$

This parameter is analogous to the inverse Knudsen number of low-density fluid mechanics, where it is recalled that the Knudsen number is the ratio of the molecular collision mean free path to the appropriate characteristic flow length.

Two limits can be distinguished immediately. In one limit, the density is so low that the photon mean free path (for all wavelengths) is large enough that the gas may be considered optically thin ($\tau \ll 1$). Under this condition, the net radiative heat transfer rate between the walls is determined simply by the blackbody flux from the walls and is given by

$$-q = \sigma(T_2^4 - T_1^4) \qquad \tau \ll 1 \tag{2}$$

Here T_2 is the absolute temperature of the hot plate, T_1 that of the cold plate, and σ is the Stefan-Boltzmann constant.

In the opposite limit of an optically thick gas ($\tau \gg 1$), the density is so large that the photons are trapped in the gas. The photon mean free path is then very small, and radiation emitted at a point is absorbed at a distance comparable to

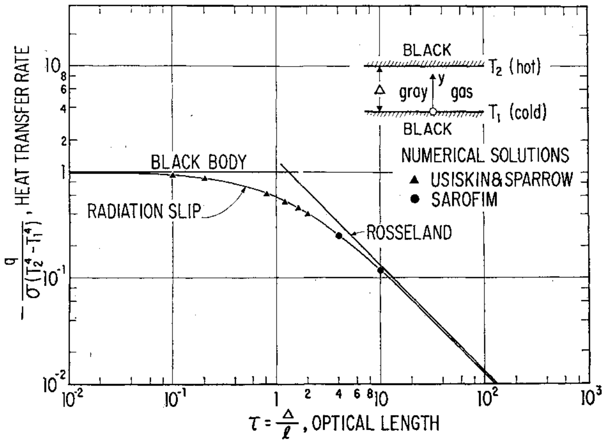

Fig. 1 Radiative heat transfer rate between two infinite isothermal parallel black plates containing a gray gas in the gap between the plates

the mean free path. As a result of the presence of the absorbing-emitting medium between the plates, there must be a reduction in the heat flux from that given by the blackbody emission of Eq. (2). For large optical thickness, the local radiant heat flux is equal to the local temperature gradient multiplied by a photon thermal conductivity k_ν. This value of the photon conductivity can be found from the relation

$$k_\nu \nabla T = (h\nu)(c/3)l_\nu \nabla n_\nu \tag{3}$$

which is obtained by direct analogy with simple kinetic theory mean free path arguments.[3] Here h is Planck's constant, ν the photon frequency, and c the speed of light. The gradient of the equilibrium photon density ∇n_ν may be written as $(dn_\nu/dT)\nabla T$. For simplicity of presentation, the gas will be taken to be gray, which implies that the photon mean free path is independent of wavelength. It should be pointed out that we anticipate that frequency-dependent mean free paths (nongray gases) also might be accounted for by similar arguments. Inserting in Eq. (3) the Planck equilibrium photon density $8\pi\nu^2/c^3[\exp(h\nu/kT) - 1]$ and integrating over all frequencies, one obtains the classical value for the radiation conductivity:[3]

$$k_R = \tfrac{16}{3}\sigma T^3 l \tag{4}$$

In the present problem, the local rate of heat transfer then is given by

$$-q = k_R(dT/dy) \tag{5}$$

with y measured from the cold plate. To calculate the net heat transfer to the surface, Eq. (5) is integrated across the gap

$$-q = \frac{1}{\Delta}\int_{T_1}^{T_2} \frac{16}{3}\,\sigma T^3 l\, dT \tag{6}$$

Again to simplify the presentation, it will be assumed further that the photon mean free path is uniform. For moderate τ, since the temperature of the gas is nearly uniform over a large part of the gap,[4] the approximation is a reasonable one. The result is that in the Rosseland limit, which will be denoted as the no-slip limit,

$$-q = \tfrac{4}{3}(\sigma/\tau)(T_2^4 - T_1^4) \qquad \tau \gg 1 \tag{7}$$

As pointed out at the beginning, to solve the intermediate range of opacity, we assume a temperature discontinuity at the plate surfaces of the form

$$T - T_w = Kl(dT/dy) \tag{8}$$

Here T_w is the wall temperature of the surface being considered, whereas T, l, and dT/dy are taken to correspond to conditions in the gas at the slip interface. The parameter K is a constant that depends on the surface conditions, among other things. Its value (assuming the nature of the surface) may be determined from a simple mean free path transport argument similar to that used in calculating the radiation conductivity. However, K will be determined here by requiring that, in the limit of the optically thin gas, the radiation conduction solution obtained with the slip boundary condition match the blackbody result given by Eq. (2).

Since the calculation of the net heat transfer rate by means of Eq. (5), subject to the boundary condition given by Eq. (8), is identical with the ordinary molecular conduction calculation (see, e.g., Kennard[5]), only the result will be written down, which is

$$q(\text{slip}) = \frac{q(\text{no slip})}{1 + 2K/\tau} \tag{9}$$

In the limit of $\tau \ll 1$, one finds from Eqs. (2, 7, and 9) that

$$K = \tfrac{2}{3} \tag{10}$$

It follows that the heat transfer rate in the intermediate range of opacities is given by the relation

$$-q = \frac{\sigma(T_2^4 - T_1^4)}{1 + 3\tau/4} \tag{11}$$

This simple result provides a complete solution for all opacities and takes on the correct limiting values for large and small τ. It is remarkable that Eq. (11) is exactly the analytic solution obtained by Lick[12] using the Milne-Eddington approximation.[6, 7]

It should be pointed out that there is little theoretical justification in extending Eq. (9) to very small values of τ. In analogous molecular problems, however, similar extrapolations show good agreement both with more refined theories and with experiments, at least for gross integrated fluid properties such as skin friction and heat transfer (see, e.g., Sherman and Talbot[8]).

On Fig. 1 the blackbody and Rosseland limit heat transfer rates have been plotted, as well as the heat transfer rate with radiation slip as given by Eq. (11). Also shown on this figure are some numerical results of Usiskin and Sparrow[4] and Sarofim.[9] These data were obtained by numerical iteration of the appropriate integral equations for radiative transport alone. It certainly is clear that the agreement is excellent.

For the example considered, conductive heat transfer can be included in a rather simple manner. It is noted that the local conductive heat transfer rate is given by Eq. (5), with the radiation conductivity k_R replaced by the ordinary molecular conduction coefficient k_c, which, for simplicity, is taken to be constant. Now both heat transfer rates obey the same type of differential relation and are integrable separately with respect to temperature. Therefore, for this problem, within the approximation of this method for representing the radiant heat transfer, the combined total net heat transfer rate may be obtained simply by adding the separate contributions due to radiation and due to conduction. Here, the radiation slip condition is applied to the radiative contribution, and the no-slip condition is applied to the conductive contribution. With conduction present, of course, there no longer can be any physical temperature discontinuity at the plates as long as the molecular mean free path is small. However, because of the separability of the equations and their integrals, the net heat transfer rate can be calculated as two separate heat fluxes in parallel. This shows, then, that, at least in this simple case, the photon gas behaves independently of the molecular gas.

Integrating and adding the separate contributions as indicated gives for the net heat transfer rate

$$-q = \frac{\sigma(T_2^4 - T_1^4)}{1 + 3\tau/4} + k_c\,\frac{(T_2 - T_1)}{\Delta} \tag{12}$$

With $T_1^4 \ll T_2^4$, this relation may be re-expressed in the form

$$-q/\sigma T_2^4 = [1 + 3\tau/4]^{-1} + \tfrac{4}{3}(N/\tau)(1 - T_1/T_2) \tag{13}$$

where N is proportional to the ratio of the molecular to radiation conductivity

$$N = 4\,\frac{k_c}{k_R} = \frac{k_c}{\tfrac{4}{3}\sigma T_2^3 l} \tag{14}$$

The parameter N is then a measure of the importance of conductive heat transfer in comparison with radiative transfer.

On Fig. 2, the heat transfer rate given by Eq. (13) has been plotted for a wall temperature ratio $T_1/T_2 = 0.1$, along with the pure radiation and pure conduction limits corresponding to $N \ll 1$ and $N \gg 1$, respectively. Also shown on this figure are numerical results of Viskanta and Grosh[10] and of Lick[12] obtained by solving the governing integro-differential equation. The present analytic solution is seen to be in good

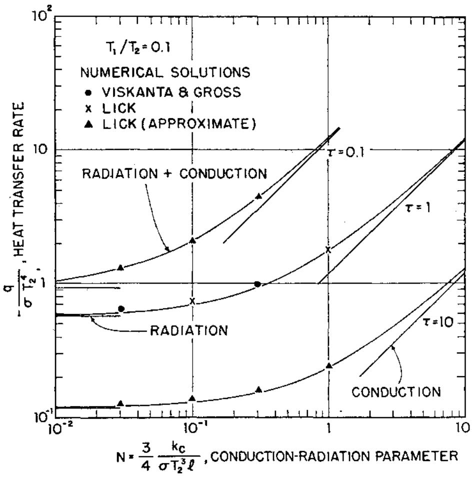

Fig. 2 Heat transfer rate for simultaneous radiation and conduction between two infinite isothermal parallel black plates containing a gray, constant conductivity gas in the gap between the plates

agreement with both the exact numerical calculations and the approximate calculations of Lick obtained using various asymptotic expansions.

In view of the success of the radiation slip argument, at least for the particular example considered, it would appear worth while to apply this concept to more complicated geometries, including problems where molecular conduction is present simultaneously, and to gas flows where radiant heat transfer is of importance (see, e.g., Goulard[11]). It also would appear that the exploitation of the simple mean free path arguments presented, by a more detailed photon transport theory, could assist materially in the difficult calculations of radiation transition regimes where the surfaces present and the radiating medium have more complicated properties than those considered.

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Reduction of Frozen Flow Losses by Nonequilibrium Heating

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Introduction

THE energy absorbed in the dissociation of a polyatomic propellant in nuclear or electrothermal rockets often represents a large portion of the total energy input. Current information on recombination rates appears to preclude any significant recombination of these dissociated particles in a nozzle of reasonable length. Thus the unreclaimed dissociation energy constitutes a serious loss mechanism. This energy loss, called the frozen flow loss, has been estimated for various propellants,[1] usually with the assumption that the propellant is dissociated to equilibrium compositions corresponding to the stagnation or throat enthalpies. This type of assumptions leads to the well-known conclusion that in order to reduce frozen flow losses one should operate at as high a pressure as practicable. Small improvements in frozen flow loss due to temperature nonuniformities and regenerative heating also have been investigated.[2–4]

All these studies have neglected the problem of dissociation kinetics. As will be indicated below, under certain circumstances, it is possible to heat the propellant to high enthalpies rapidly without significant dissociation. These preliminary calculations indicate that nonequilibrium heating is a promising means for reducing frozen flow losses of high specific impulse engines.

Discussion

The dissociation reaction rate is usually comparable to the recombination rate. Therefore the assumption of equilibrium dissociation but nonequilibrium recombination in rocket nozzle calculations is valid only for nozzles where the propellant is heated in low velocity or essentially stagnant conditions. If a nozzle is constructed in such a manner that the propellant is heated and simultaneously accelerated to high speeds, dissociation equilibrium may not be satisfied. At sufficiently high heating rates and acceleration rate, it is conceivable that the heated propellant may be accelerated before significant dissociation takes place.

In order to obtain an order of magnitude estimate of the feasibility of nonequilibrium heating, one may compare the time required for dissociation to the characteristic time of the nozzle flow as well as the characteristic times of the pertinent heating processes. The dissociation time constant is approximately

$$\tau_d \approx 1/\beta N_m = 1/\alpha K_N N_m \tag{1}$$

where the relationship between the dissociation coefficient and recombination coefficient has been used. Based on Rink's summary of recent data[5] and Keck's theoretical temperature dependence,[6] α can be taken approximately as $8\,(10)^{-30}\,T^{-1}$ cc^2/molecule^2-sec. The result of the calculation is shown in Table 1 as a function of pressure.

The characteristic residence time of the propellant travelling at near sonic or higher velocities in a nozzle is related to the gas enthalpy and the nozzle length. Neglecting dissociation

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